Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **39**, 1–9; doi: 10.14232/ejqtde.2015.1.39 http://www.math.u-szeged.hu/ejqtde/

L_1-maximal regularity for quasilinear second order differential equation with damped term

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Received 7 March 2015, appeared 13 July 2015
Communicated by Michal Fečkan

Abstract. We investigated a quasilinear second order equation with damped term on the real axis. We gave some suitable conditions for existence of the L_1-maximal regular solutions of this equation.

Keywords: differential equation, nonlinear damped term, separation of operator, fixed point theorem.

2010 Mathematics Subject Classification: 34A34.

1 Introduction and statement of main result

Let $\mathbb{R} := (-\infty, +\infty)$, $L_1 := L_1(\mathbb{R})$ and $\|\cdot\|_1$ be the norm of L_1. We consider the equation

$$\mathbf{L}y := -y'' + r(x,y)y' + q(x,y)y = f(x), \qquad x \in \mathbb{R}, \tag{1.1}$$

where r is continuously differentiable and q is a continuous function, $f \in L_1$. This is a useful equation in mathematical physics (see [5, 26]).

By $C_0^{(k)}(\mathbb{R})$ $(k = 1, 2, \dots)$ we denote the set of k times continuously differentiable functions with compact support. Let $C_{\mathrm{loc}}^{(j)}(\mathbb{R}) = \{y : \psi y \in C_0^{(j)}(\mathbb{R}),\ \forall \psi \in C_0^{(\infty)}(\mathbb{R})\}$ $(j = 1, 2)$.

Definition 1.1. Let $y \in L_1$, if there is a sequence $\{y_n\}_{n=1}^{\infty} \subset C_{\mathrm{loc}}^{(2)}(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|\psi(y_n - y)\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\psi(\mathbf{L}y_n - f)\|_1 = 0,$$

$\forall\, \psi \in C_0^{(\infty)}(\mathbb{R})$. Then y is called a solution of (1.1).

The purpose of this work is to find some conditions for r and q such that for every $f \in L_1$, the equation (1.1) has a solution y which satisfies

$$\|y''\|_1 + \|r(\cdot,y)y'\|_1 + \|q(\cdot,y)y\|_1 < \infty.$$

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The separability of differential operators introduced by Everitt and Giertz in [7, 8] plays an important role in the study of second order differential equations. Recall that the Sturm–Liouville operator

$$\widetilde{L}y := -y'' + q_1(x)y,$$

acting in $L_2(\mathbb{R})$ is separable, if there is a constant $c > 0$ such that

$$\| -y''\|_2 + \|q_1(\cdot)y\|_2 \le c(\|\widetilde{L}y\|_2 + \|y\|_2), \qquad \forall\, y \in D(\widetilde{L}).$$

Everitt and Giertz [7, 8] proved that if q_1 and its derivatives satisfy some conditions, then $\widetilde{L}$ is separable in $L_2(\mathbb{R})$. In the case q_1 is not differentiable function, the separability of $\widetilde{L}$ in $L_2(\mathbb{R})$ was discussed in [3, 23]. In [9], Everitt, Giertz and Weidmann give an example of non-separable Sturm–Liouville operator in $L_2(\mathbb{R})$ with strongly oscillating and infinitely smooth coefficient q_1. The separability of linear partial differential operators was studied in [4, 15, 17, 21, 24]. Some sufficient conditions of separability of operators on Riemann manifolds are obtained in [1, 2, 12, 13].

The separability is also an important tool when dealing with quasilinear equations. In [16], Muratbekov and Otelbaev used the separability to discuss the solvability of the nonlinear equation

$$-y'' + q_0(x,y)y = f(x), \tag{1.2}$$

where $f \in L_2(\mathbb{R})$. Grinshpun and Otelbaev showed that the solvability of the equation (1.2) in L_1 implies $q_0 \ge 1$ (see [6]). This method is useful for the multidimensional (Schrödinger) equation $-\Delta u + q(x,\, u)u = F(x),\ x \in \mathbb{R}^n$ (see [17, 20] for details).

In general, the expression (1.1) can be converted neither to (1.2) nor to the form

$$-\left(p_2(x,y)y'\right)' + q_2(x,y)y = f(x).$$

In [22], we considered the equation $-y'' + r(x,y)y' = f(x)$, $f \in L_2(\mathbb{R})$, and found some conditions for r such that this equation is solvable. In the present paper, we discuss the more general equation (1.1), in the case $f \in L_1$. Under weaker conditions on r than in [22] the existence and regularity of solutions of (1.1) are established.

Schauder's fixed-point theorem is used to prove our main result (see [10]).

Let g and h be some functions on $\mathbb{R}$ and let

$$\alpha_{g,h}(t) = \int_0^t |g(\eta)|\,d\eta \operatorname*{ess\,sup}_{\xi\in(t,+\infty)} |h(\xi)|^{-1}, \qquad t > 0,$$

$$\beta_{g,h}(\tau) = \int_\tau^0 |g(\eta)|\,d\eta \operatorname*{ess\,sup}_{\xi\in(-\infty,\tau)} |h(\xi)|^{-1}, \qquad \tau < 0,$$

$$\gamma_{g,h} = \max\left(\sup_{t>0}\alpha_{g,h}(t),\, \sup_{\tau<0}\beta_{g,h}(\tau)\right).$$

The main result of this paper is the following.

Theorem 1.2. *Let r be a continuously differentiable function and q a continuous function satisfying*

$$r \ge \delta_1\sqrt{1+x^2} \qquad (\delta_1 > 0) \tag{1.3}$$

and

$$\sup_{c_0\in\mathbb{R}} \gamma_{q(\cdot,c_0),r(\cdot,c_0)} < \infty. \tag{1.4}$$

Then for any $f \in L_1$, the equation (1.1) *has a solution y such that*

$$\|y''\|_1 + \|r(\cdot, y)y'\|_1 + \|q(\cdot, y)y\|_1 < \infty. \tag{1.5}$$

Example 1.3. Let $r = 10 + x^{10} + 5y^4$, $q = x^3 + \cos^4 x + 2y$. Then r and q satisfy the conditions of Theorem 1.2.

2 Auxiliary statements

By Muckenhoupt's theorem (Theorem 2 in [14]), we obtain the following lemma.

Lemma 2.1. *Let g and h be continuous functions on $\mathbb{R}$ such that $\gamma_{g,h} < \infty$. Then*

$$\int_{\mathbb{R}} |g(x)y(x)|\, dx \le \gamma_{g,h} \int_{\mathbb{R}} |h(x)y'(x)|\, dx, \qquad \forall\, y \in C_0^{(1)}(\mathbb{R}). \tag{2.1}$$

Moreover, $\gamma_{g,h}$ is the smallest constant which satisfies (2.1).

Let r be a continuously differentiable function and

$$l_0 y = -y'' + r(x)y', \qquad D(l_0) = C_0^{(2)}(\mathbb{R}).$$

Denote by l the closure of l_0 in L_1.

Lemma 2.2. *Let r be continuously differentiable and satisfy* (1.3). *Then l is invertible, $R(l) = L_1$ and*

$$\|y''\|_1 + \|ry'\|_1 \le 3\|ly\|_1, \qquad \forall\, y \in D(l). \tag{2.2}$$

Proof. We use the method of [25] to prove (2.2). Let $y \in C_0^{(2)}(\mathbb{R})$ be a real function, and $\gamma > -1$. Then by integration by parts, we get

$$\int_{\mathbb{R}} (ly)y' \left[(y')^2\right]^{\gamma/2} dx = \int_{\mathbb{R}} r \left[(y')^2\right]^{\gamma/2+1} dx.$$

By Hölder's inequality, we have that

$$\int_{\mathbb{R}} r \left[(y')^2\right]^{\gamma/2+1} dx \le \left[\int_{\mathbb{R}} |r^{-\alpha} ly|^p\, dx\right]^{1/p} \left[\int_{\mathbb{R}} |r^{\alpha}(y')^{\gamma+1}|^q\, dx\right]^{1/q}, \tag{2.3}$$

where $1 < p < \infty$, $q = p/(p-1)$. We choose α and γ as follows: $(\gamma+1)\,q = \gamma + 2$, $\alpha q = 1$. This implies $\gamma + 2 = p$. So from (2.3) it follows that $\|r^{1/p}y'\|_p \le \|r^{-1/q}ly\|_p$. Taking the limit $p \to 1$, we get

$$\|ry'\|_1 \le \|ly\|_1, \qquad y \in C_0^{(2)}(\mathbb{R}). \tag{2.4}$$

Since $\|y''\|_1 \le \|ly\|_1 + \|ry'\|_1 \le 2\|ly\|_1$, we have $\|y''\|_1 + \|ry'\|_1 \le 3\|ly\|_1$, $\forall\, y \in C_0^{(2)}(\mathbb{R})$. Since l is a closed operator, the last inequality holds for any $y \in D(l)$.

From Lemma 2.1, (1.3) and (2.4) follows that

$$\|y\|_1 \le c_1\|ly\|_1, \qquad y \in D(l). \tag{2.5}$$

So the inverse l^{-1} of l exists.

Next, we show that $R(l) = L_1$. Let $R(l) \neq L_1$. Since l is closed, (2.5) implies $R(l)$ is closed. Hence there exists a nonzero element $z_0 \in {}^{\perp}R(l)$ such that $l^* z_0 = -z_0'' - (r(x)z_0)' = 0$, where l^* is adjoint operator of l. Then

$$\left[z_0 \exp\left(\int_a^x r(\eta)\,d\eta\right)\right]' = c_2 \exp\left(\int_a^x r(\eta)\,d\eta\right).$$

Let $c_2 \neq 0$. Without loss of generality, we can assume that $c_2 = -1$. Then

$$\left(z_0 \exp\left(\int_a^x r(\eta)\,d\eta\right)\right)' < 0,$$

i.e. $z_0(x)$ is a monotonically decreasing function, moreover $z_0(x-k) > \exp(k) z_0(x)$, $x \in \mathbb{R}$, $k = 1, 2, \ldots$, which implies that $z_0 \notin L_\infty(\mathbb{R})$.

Let $c_2 = 0$. Then

$$z_0 = c_3 \exp\left(-\int_a^x r(\eta)\,d\eta\right), \qquad c_3 \neq 0.$$

Therefore $z_0 \notin L_\infty(\mathbb{R})$. We obtain a contradiction, hence $R(l) = L_1$. □

We consider the following linear equation

$$ly := -y'' + r_1(x)y' + q_1(x)y = f(x), \qquad x \in \mathbb{R}. \tag{2.6}$$

The function $y \in L_1$ is called a solution of (2.6), if there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{(2)}(\mathbb{R})$ such that $\|y_n - y\|_1 \to 0$ and $\|ly_n - f\|_1 \to 0$ as $n \to \infty$.

Lemma 2.3. *Let r_1 be a continuously differentiable function such that $r_1 \geq \delta_1\sqrt{1+x^2}$. Assume q_1 is a continuous function and $\gamma_{q_1,r_1} < \infty$. Then for every $f \in L_1$, the equation (2.6) has a unique solution y such that*

$$\|y''\|_1 + \|r_1 y'\|_1 + \|q_1 y\|_1 \leq c_4 \|f\|_1, \tag{2.7}$$

where c_4 depends only on γ_{q_1,r_1}.

Proof. Let $x = at$ $(a > 0)$, then (2.6) becomes that

$$-\tilde{y}'' + a^{-1}\tilde{r}_1(t)\tilde{y}' + a^{-2}\tilde{q}_1(t)\tilde{y} = \tilde{f}, \tag{2.8}$$

where $\tilde{y}(t) = y(at)$, $\tilde{r}_1(t) = r_1(at)$, $\tilde{q}_1(t) = q_1(at)$, $\tilde{f}(t) = a^{-2} f(at)$. Let $l_{0a}\tilde{y} = -\tilde{y}_{tt}'' + a^{-1}\tilde{r}_1\tilde{y}'$, $\tilde{y} \in C_0^{(2)}(\mathbb{R})$. By l_a we denote the closure in L_1 of l_{0a}. Since $a^{-1}\tilde{r}_1(t)$ satisfies the conditions of Lemma 2.2, it follows that the operator l_a is continuously invertible and

$$\|\tilde{y}''\|_1 + \|a^{-1}\tilde{r}_1\tilde{y}'\|_1 \leq 3\|l_a\tilde{y}\|_1, \qquad \forall\, \tilde{y} \in D(l_a). \tag{2.9}$$

Let $a = 4\left(1 + \gamma_{\tilde{q}_1,\tilde{r}_1}\right)$. By 2.1, we obtain that

$$\|a^{-2}\tilde{q}_1\tilde{y}\|_1 \leq \frac{\gamma_{\tilde{q}_1,\tilde{r}_1}}{a^2}\|\tilde{r}_1\tilde{y}'\|_1 \leq \frac{3}{4}\|l_a\tilde{y}\|_1, \qquad \forall\, \tilde{y} \in D(l_a). \tag{2.10}$$

Hence, by a well-known theorem (see [11, Chapter 4, Theorem 1.16]), we find that the operator $l_a + a^{-2}\tilde{q}_1(t)E$ corresponding to (2.8) is invertible and $R\left(l_a + a^{-2}\tilde{q}_1 E\right) = L_1$. Let $\tilde{y}$ be a solution of the equation (2.8), by (2.9) and (2.10), we obtain that

$$\|\tilde{y}''\|_1 + \|a^{-1}\tilde{r}_1\tilde{y}'\|_1 + \|a^{-2}\tilde{q}_1\tilde{y}\|_1 \leq 4\|l_a\tilde{y}\|_1. \tag{2.11}$$

From (2.10) it follows that

$$\|a^{-2}\tilde{q}_1\tilde{y}\|_1 \le 3\| \left(l_a + a^{-2}\tilde{q}_1 E\right) \tilde{y}\|_1.$$

So

$$\|l_a\tilde{y}\|_1 \le \| \left(l_a + a^{-2}\tilde{q}_1 E\right) \tilde{y}\|_1 + \|a^{-2}\tilde{q}_1\tilde{y}\|_1 \le 4\| \left(l_a + a^{-2}\tilde{q}_1 E\right) \tilde{y}\|_1. \tag{2.12}$$

The inequalities (2.11) and (2.12) imply that for the solution $\tilde{y}$ of (2.8), the following inequality holds:

$$\|\tilde{y}''\|_1 + \|a^{-1}\tilde{r}_1\tilde{y}'\|_1 + \|a^{-2}\tilde{q}_1\tilde{y}\|_1 \le 16\|\tilde{f}\|_1.$$

Taking $t = a^{-1}x$, we to obtain (2.7). □

Remark 2.4. Let $r_1(x)$ be continuously differentiable. Assume $q_1(x)$ is a continuous function. L will denote the closure in L_1 of the operator $L_0 y := -y'' + r_1 y' + q_1 y$, $D(L_0) = C_0^{(2)}(\mathbb{R})$. If there is a constant $c_5 > 0$ such that $\| - y''\|_1 + \|r_1(\cdot)y'\|_1 + \|q_1(\cdot)y\|_1 \le c_5 \left(\|Ly\|_1 + \|y\|_1\right), \forall\, y \in D(L)$, then L is called separable in L_1.

If the conditions of Lemma 2.3 hold, then the operator L is separable in L_1.

3 Proof of the main theorem

Let $C(\mathbb{R})$ be the space of bounded continuous functions on $\mathbb{R}$ with the norm $\|y\|_{C(\mathbb{R})} = \sup_{x\in\mathbb{R}} |y(x)|$. Let ε and A be positive numbers. Set

$$S_A = \left\{ z \in C(\mathbb{R}) : \sup_{x\in\mathbb{R}} |z(x)| \le A \right\}.$$

Let $v \in S_A$. $L_{v,\varepsilon}$ denote the closure in L_1 of the following linear differential expression

$$L_{0,v,\varepsilon} y := -y'' + \left[r(x, v(x)) + \varepsilon \left(1 + x^2\right)\right] y' + q(x, v(x))y, \qquad \forall\, y \in C_0^{(2)}(\mathbb{R}).$$

We consider the equation

$$L_{v,\varepsilon} y = f(x). \tag{3.1}$$

$\tilde{r}_{1,\varepsilon,v}(x) := r(x, v(x)) + \varepsilon \left(1 + x^2\right)$ and $\tilde{q}_v(x) := q(x, v(x))$ satisfy all of the conditions of Lemma 2.3. Indeed, by (1.3), $\tilde{r}_{1,\varepsilon,v}(x) \ge \delta_1\sqrt{1 + x^2}$, $\delta_1 > 0$. Hence $\gamma_{1,\tilde{r}_{1,\varepsilon,v}} < \infty$. From (1.4) it follows that $\gamma_{\tilde{q}_v(x),\tilde{r}_{1,\varepsilon,v}(x)} \le C_1 \sup_{t\in\mathbb{R}} \gamma_{q(x,t),r(x,t)} < \infty$. Therefore, for any $f \in L_1$, the equation (3.1) has a unique solution y and

$$\|y''\|_1 + \left\| \left[r\left(\cdot\,, v(\cdot)\right) + \varepsilon(1 + x^2)\right] y' \right\|_1 + \|q\left(\cdot\,, v(\cdot)\right) y\|_1 \le C_2\|f\|_1, \tag{3.2}$$

where C_2 does not depend on A. By 2.1, we have that

$$\|y\|_1 \le C_3\|\tilde{r}_{1,\varepsilon,v} y'\|_1, \qquad \left\|\sqrt{1 + x^2} y\right\|_1 \le C_4 \left\|\left(1 + x^2\right) y'\right\|_1. \tag{3.3}$$

By using (3.2), (3.3) and Theorem 1 given in Chapter 3 of [18], we obtain that

$$\begin{aligned} \|y\|_W := \|y''\|_1 + \left\|\left[r(\cdot, v(\cdot)) + (1 + x^2)\right] y'\right\|_1 + \left\|\left[|q(\cdot, v(\cdot))| + \sqrt{1 + x^2}\right] y\right\|_1 \\ + \sup_{x\in\mathbb{R}} \left|(1 + x^2)^{3/8} y(x)\right| \le C_5\|f\|_1, \qquad y \in D(L_{v,\varepsilon}), \end{aligned} \tag{3.4}$$

where C_5 also does not depend on A.

Let $A = C_5\|f\|_1 + 1$. Set $P_\varepsilon(v) = L_{v,\varepsilon}^{-1} f$, where $v \in S_A$, $\varepsilon > 0$, $f \in L_1$ and $L_{v,\varepsilon}^{-1}$ is inverse to $L_{v,\varepsilon}$. According to (3.4), the operator P_ε maps S_A into itself. Moreover, the operator P_ε maps S_A to the set

$$Q_A := \{y : \|y\|_W \le C_5\|f\|_1\}.$$

Q_A is compact in $C(\mathbb{R})$. Indeed, let $\gamma > 0$, then by (3.4) there exists $l \in \mathbb{N}$ such that for any $z \in Q_A$

$$\begin{aligned} &\|z''\|_{L_1(\mathbb{R}\setminus[-l,l])} + \left\| \left[r(\cdot, v(\cdot)) + (1+x^2) \right] z' \right\|_{L_1(\mathbb{R}\setminus[-l,l])} \\ &\qquad + \left\| \left[|q(\cdot, v(\cdot))| + \sqrt{1+x^2} \right] z \right\|_{L_1(\mathbb{R}\setminus[-l,l])} < \gamma/2, \end{aligned} \tag{3.5}$$

and

$$\sup_{x:|x|\ge l} \frac{C_5\|f\|_1 + 1}{(1+x^2)^{3/8}} < \gamma/2. \tag{3.6}$$

Let $\varphi_l \in C_0^{(\infty)}(-l-1, l+1)$ $(l = 1, 2, \ldots)$ such that $\varphi_l(x) = 1$ for $x \in [-l, l]$, $\varphi_l(x) = 0$ for $x \notin [-l-1, l+1]$ and $0 \le \varphi_l \le 1$. We denote $T_l = \{\varphi_l z : z \in Q_A\}$. By (3.5) and (3.6), T_l is a γ-net of Q_A. On the other hand, T_l is a subset of the Sobolev space

$$\mathring{W}_1^2(-l-1, l+1) = \left\{ \theta \in W_1^2(-l-1, l+1) : \theta(x) = 0, \text{ as } |x| \ge l+1 \right\}.$$

Notice that the embedding of $\mathring{W}_1^2(-l-1, l+1)$ in $C_0[-l-1, l+1]$ is compact, where $C_0[-l-1, l+1] = \{\eta \in C[-l-1, l+1] : \eta(x) = 0, \text{ as } |x| \ge l+1\}$ (see [19, 27]). So T_l is a compact γ-net of Q_A. By Hausdorff's theorem (see [10, Chapter 1]), Q_A is compact in $C(\mathbb{R})$.

Next, we show that the operator P_ε is continuous on S_A. Let $\{v_n\}_{n=1}^\infty \subset S_A$ be a sequence such that $\sup_{x\in\mathbb{R}} |v_n(x) - v(x)| \to 0$ as $n \to +\infty$. If y_n $(n = 1, 2, \ldots)$ and y satisfy

$$L_{v,\varepsilon} y = f, \qquad L_{v_n,\varepsilon} y_n = f. \tag{3.7}$$

Then $P_\varepsilon(v_n) - P_\varepsilon(v) = y_n - y$. So it suffices to show that $\sup_{x\in\mathbb{R}} |y_n(x) - y(x)| \to 0$ as $n \to \infty$. By (3.7), we deduce that

$$y - y_n = L_{v,\varepsilon}^{-1} \left\{ \left[r(x, v_n(x)) - r(x, v(x)) \right] y_n' + \left[q(x, v_n(x)) - q(x, v(x)) \right] y_n \right\}. \tag{3.8}$$

Since functions v and v_n $(n = 1, 2, \ldots)$ are continuous, we see that $r(x, v_n(x)) - r(x, v(x))$ and $q(x, v_n(x)) - q(x, v(x))$ are continuous functions. Therefore, from (3.8) it follows that

$$\begin{aligned} \|y_n - y\|_{L_1(-a,a)} \le c \max_{x\in[-a,a]} &\left[r(x, v_n(x)) - r(x, v(x))|, |q(x, v_n(x)) - q(x, v(x))| \right] \\ &\times \left[\|y_n'\|_{L_1(-a,a)} + \|y_n\|_{L_1(-a,a)} \right] \to 0 \end{aligned} \tag{3.9}$$

as $n \to \infty$, for every $a > 0$. On the other hand, by (3.4), we have $\{y_n\}_{n=1}^{+\infty} \subset Q_A$, $\|y_n\|_W \le A$ $(n = 1, 2, \ldots)$, $y \in Q_A$, $\|y\|_W \le A$. Since the set Q_A is compact in $C(\mathbb{R})$, without loss of generality, we can assume that the sequence $\{y_n\}_{n=1}^{+\infty}$ converges to some $z \in C(\mathbb{R})$. By (3.4)

$$\lim_{|x|\to\infty} y(x) = 0, \qquad \lim_{|x|\to\infty} z(x) = 0. \tag{3.10}$$

Since the operator $L_{v,\varepsilon}^{-1}$ is closed, by (3.9) and (3.10), we obtain that $z = y$. Thus P_ε is continuous.

So P_ε is a completely continuous operator in $C(\mathbb{R})$ and it maps the ball S_A into itself. By Schauder's theorem (see [10, Chapter XVI]), P_ε has a fixed point y in S_A, i.e. $P_\varepsilon(y) = y$. And y satisfies the equality

$$-y'' + \left[r(x, y) + \varepsilon(1+x^2)\right] y' + q(x,y)y = f(x).$$

By Lemma 2.3, we obtain that

$$\|y''\|_1 + \|\left[r(\cdot, y) + \varepsilon(1+x^2)\right] y'\|_1 + \|q(\cdot, y)\, y'\|_1 \le C_5\|f\|_1.$$

Now, let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that $\lim_{j\to\infty}\varepsilon_j = 0$. Recall that $P_{\varepsilon_j}(v) = L^{-1}_{v,\varepsilon_j} f$. If $y_j \in S_A$ is the fixed point of the operator P_{ε_j}, then

$$-y_j'' + \left[r(x, y_j) + \varepsilon_j(1+x^2)\right] y_j' + q(x,y_j)y_j = f(x).$$

Then according to Lemma 2.3, we have

$$\left\|y_j''\right\|_1 + \left\|\left[r(\cdot, y_j(\cdot)) + \varepsilon_j\,(1+x^2)\right]y_j'\right\|_1 + \left\|q(\cdot, y_j(\cdot))\, y_j\right\|_1 \le C_5\|f\|_1. \tag{3.11}$$

Let (a,b) be an arbitrary finite interval. It is known that the space $W_1^2(a, b)$ is compactly embedded to $L_1(a, b)$. Therefore, by virtue of (3.11), we can select a subsequence $\{\tilde{y}_j\}_{j=1}^{\infty}$ of $\{y_j\}_{j=1}^{\infty} \subset W_1^2(a, b)$ such that $\|\tilde{y}_j - y\|_{L_1(a,b)} \to 0$ as $j \to \infty$. By Definition 1.1, y is a solution of equation (1.1). By Lemma 2.3, we obtain that for y the estimate (1.5) holds. □

Remark 3.1. The condition (1.3) is natural. If (1.3) does not hold, from Lemma 2.1 it follows that the domain $D(\mathbf{L})$ of $\mathbf{L}$ is not included in L_1.

Acknowledgements

This work is supported by project 5132/GF4 of Science Committee of Ministry of Education and Science of Republic of Kazakhstan.

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